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MODERATE SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS WITH HARDY POTENTIAL UNDER MINIMAL RESTRICTIONS ON THE POTENTIAL

MOSHE MARCUS AND VITALY MOROZ

ABSTRACT. We study semilinear elliptic equations with Hardy potential
 $(E) \quad -\mathcal{L}_\mu u + u^q = 0$

in a bounded smooth domain $\Omega \subset \mathbb{R}^N$. Here $q > 1$, $\mathcal{L}_\mu = \Delta + \frac{\mu}{\delta_\Omega^2}$ and $\delta_\Omega(x) = \text{dist}(x, \partial\Omega)$. Assuming that $0 \leq \mu < C_H(\Omega)$, boundary value problems with measure data and discrete boundary singularities for positive solutions of (E) have been studied in [10]. In the case of *convex* domains $C_H(\Omega) = 1/4$. In this case similar problems have been studied in [8]. In the present paper we study these problems, in *arbitrary domains*, assuming only $-\infty < \mu < 1/4$, even if $C_H(\Omega) < 1/4$. We recall that $C_H(\Omega) \leq 1/4$ and, in general, strict inequality holds. The key to our study is the fact that, if $\mu < 1/4$ then in smooth domains there exist local \mathcal{L}_μ -superharmonic functions in a neighborhood of $\partial\Omega$ (even if $C_H(\Omega) < 1/4$). Using this fact we extend the notion of *normalized boundary trace* introduced in [10], to arbitrary domains, provided that $\mu < 1/4$. Further we study the b.v.p. with normalized boundary trace ν in the space of positive finite measures on $\partial\Omega$. We show that existence depends on two critical values of the exponent q and discuss the question of uniqueness. Part of the paper is devoted to the study of the linear operator: properties of local \mathcal{L}_μ -subharmonic and superharmonic functions and the related notion of moderate solutions. Here we extend and/or improve results of [5] and [10] which are later used in the study of the nonlinear problem.

1. INTRODUCTION AND MAIN RESULTS

1.1. Introduction. On bounded smooth domains $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) we study semilinear elliptic equations with Hardy potential of the form,

$$(P_\mu) \quad -\Delta u - \frac{\mu}{\delta_\Omega^2} u + |u|^{q-1} u = 0 \quad \text{in } \Omega,$$

where $q > 1$, $-\infty < \mu < 1/4$ and

$$\delta_\Omega(x) := \text{dist}(x, \partial\Omega).$$

Equations (P_0) had been extensively studied in the past two decades and by now the structure of the set of positive solution of such equations is well understood, see [11] and further references therein. Equation (P_μ) with Hardy potential, i.e. with $\mu \neq 0$, had been first considered in [5], where a

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classification of positive solutions had been introduced and conditions for the existence and nonexistence of *large* solutions for (P_μ) had been derived.

The study and classification of positive solutions of equation (P_μ) relies on the properties of the associated linear equation

$$(1.1) \quad -\mathcal{L}_\mu h = 0 \quad \text{in } \Omega,$$

where

$$\mathcal{L}_\mu := \Delta + \frac{\mu}{\delta_\Omega^2}.$$

Denote

$$\alpha_\pm := \frac{1}{2} \pm \sqrt{\frac{1}{4} - \mu}$$

and note that $\alpha_+ + \alpha_- = 1$. For $\rho > 0$ and $\varepsilon \in (0, \rho)$ we use the notation

$$\begin{aligned} \Omega_\rho &:= \{x \in \Omega : \delta(x) < \rho\}, & \Omega_{\varepsilon, \rho} &:= \{x \in \Omega : \varepsilon < \delta(x) < \rho\} \\ D_\rho &:= \{x \in \Omega : \delta(x) > \rho\}, & \Sigma_\rho &:= \{x \in \Omega : \delta(x) = \rho\}. \end{aligned}$$

A function $w \in L_{\text{loc}}^1(G)$ is a \mathcal{L}_μ -subharmonic in Ω if $\mathcal{L}_\mu w \leq 0$ in the distribution sense, i.e.,

$$\int_G w(-\Delta\varphi) dx - \int_G \frac{\mu}{\delta_\Omega^2} w\varphi dx \leq 0 \quad \forall 0 \leq \varphi \in C_c^\infty(\Omega).$$

We say that w is a *local* \mathcal{L}_μ -subharmonic function if there exists $\rho > 0$ such that $w \in L_{\text{loc}}^1(\Omega_\rho)$ is subharmonic in Ω_ρ . Similarly, (local) \mathcal{L}_μ -superharmonic functions are defined with “ \geq ” in the above inequality.

1.2. The role of the Hardy constant. The existence and properties of positive \mathcal{L}_μ -harmonic and superharmonic functions in Ω are controlled by the Hardy constant of the domain, defined as

$$(1.2) \quad C_H(\Omega) := \inf_{C_c^\infty(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega \frac{u^2}{\delta_\Omega^2} dx}.$$

For a bounded Lipschitz domain it is known that $C_H(\Omega) \in (0, 1/4]$. If Ω is convex then $C_H(\Omega) = 1/4$. In general, $C_H(\Omega)$ varies with the domain and could be arbitrary small (see, e.g. [9, Theorem I and Section 4]) for a discussion and examples).

Denote the *local* Hardy constant in Ω_ρ relative to $\partial\Omega$ by

$$(1.3) \quad C_H^{\partial\Omega}(\Omega_\rho) := \inf_{C_c^\infty(\Omega_\rho) \setminus \{0\}} \frac{\int_{\Omega_\rho} |\nabla u|^2 dx}{\int_{\Omega_\rho} \frac{u^2}{\delta_\Omega^2} dx}.$$

Note the difference between $C_H^{\partial\Omega}(\Omega_\rho)$ and $C_H(\Omega_\rho)$: the distance involved in the first one is $\delta_\Omega(x) = \text{dist}(x, \partial\Omega)$ while in the second it is $\delta_{\Omega_\rho}(x) = \text{dist}(x, \partial\Omega_\rho)$. Obviously $C_H^{\partial\Omega}(\Omega_\rho) \geq C_H(\Omega_\rho)$.

The following lemma shows that in contrast to the “global” Hardy constant $C_H(\Omega)$ the value of the “local” Hardy constant $C_H^{\partial\Omega}(\Omega_\rho)$ does not depend on the shape of Ω , provided that ρ is sufficiently small.

Lemma 1.1. (LOCAL HARDY INEQUALITY) *There exists $\bar{\rho} = \bar{\rho}(\Omega) > 0$ such that for every $\rho \in (0, \bar{\rho}]$ one has $C_H^{\partial\Omega}(\Omega_\rho) = C_H(\Omega_\rho) = 1/4$.*

The fact that $C_H^{\partial\Omega}(\Omega_\rho) = 1/4$ is due to [9, p.3246], while $C_H(\Omega_\rho) = 1/4$ follows from [6, Lemma 1.2].

The relation between Hardy constant and the existence of positive \mathcal{L}_μ -superharmonics is explained by the following classical result, cf. [9, p.3246].

Lemma 1.2. *Equation (1.1) admits a positive \mathcal{L}_μ -superharmonic function in Ω if and only if $\mu \leq C_H(\Omega)$.*

Equation (1.1) admits a positive \mathcal{L}_μ -superharmonic in Ω_ρ with $\rho \in (0, \bar{\rho})$ if and only if $\mu \leq 1/4$.

Thus, according to Lemma 1.1, if $C_H(\Omega) < 1/4$ then, for $\mu \in [C_H(\Omega), 1/4)$, there exist local positive \mathcal{L}_μ -superharmonic functions but no “global” positive \mathcal{L}_μ -superharmonic functions in Ω .

1.3. Moderate solutions and normalised boundary trace. In this work we study *moderate* positive solutions of nonlinear equation (P_μ) in the range $\mu < 1/4$, including negative values of μ . Recall that in the classical theory of equations (P_μ) with $\mu = 0$, moderate solution is a solution which is dominated by a positive harmonic function, cf. [11, pp.66-69]. This concept had been extended to equations (P_μ) with $0 \leq \mu < C_H(\Omega)$ in [10], where \mathcal{L}_μ -moderate solution is defined as a solution dominated by a positive \mathcal{L}_μ -harmonic function. This definition is not applicable in the range $\mu \in [C_H(\Omega), 1/4)$, when the set of positive \mathcal{L}_μ -harmonic function is empty. Therefore we modify it as follows:

Definition 1.3. A solution $u \in L_{loc}^1(\Omega)$ of equation (P_μ) is \mathcal{L}_μ -moderate if there exists a local positive \mathcal{L}_μ -harmonic function h such that $|u| \leq h$ in Ω_ρ for some $\rho \in (0, \bar{\rho}]$.

We are going to show that equation (P_μ) admits \mathcal{L}_μ -moderate solutions, with prescribed (normalized) boundary data, in the *entire* domain Ω for every $\mu < 1/4$, even when $C_H(\Omega) < 1/4$. The *existence* of a certain class of positive solutions was observed in [5, Lemma 4.15].

More specifically, we study the generalised boundary trace problem

$$(P_\mu^\nu) \quad \begin{cases} -\mathcal{L}_\mu u + |u|^{q-1}u &= 0 & \text{in } \Omega, \\ \text{tr}_{\partial\Omega}^*(u) &= \nu, \end{cases}$$

where $\mu < 1/4$, $q > 1$, $\nu \in \mathcal{M}^+(\partial\Omega)$ and $\text{tr}_{\partial\Omega}^*(u)$ denotes the *normalized boundary trace* of a positive Borel function u on $\partial\Omega$. A function $u \in L_{loc}^q(\Omega)$ is a solution of (P_μ^ν) if it satisfies the equation in the distribution sense and attains the indicated boundary data.

The concept of normalised boundary trace was introduced in [10] in order to classify positive moderate solutions of (P_μ^ν) in terms of their behaviour at the boundary, when $0 < \mu < C_H(\Omega)$.¹ It is defined as follows.

A nonnegative Borel function $u : \Omega \rightarrow \mathbb{R}$ possesses a *normalised boundary trace* $\nu \in \mathfrak{M}^+(\partial\Omega)$ if,

¹Actually, the assumption $\mu > 0$ was introduced in [10] only for simplicity: the normalised boundary trace is well-defined and the related results remain valid for any $\mu < C_H(\Omega)$.

$$(1.4) \quad \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^{\alpha_-}} \int_{\Sigma_\varepsilon} |u - \mathbb{K}_\mu^\Omega[\nu]| dS = 0$$

where K_μ^Ω is the Martin kernel of \mathcal{L}_μ in Ω . If, for a given u there exists a measure ν as above then it is unique.

By Ancona [2], if $\mu < C_H(\Omega)$ there is a (1-1) correspondence between the set of positive \mathcal{L}_μ -harmonic functions in Ω and $\mathfrak{M}^+(\partial\Omega)$; the \mathcal{L}_μ -harmonic function v corresponding to a measure ν has the representation $v = K_\mu^\Omega[\nu]$. (For details and notation see Subsection 2.1 below.)

We point out that, except in the case $\mu = 0$, $\text{tr}_{\partial\Omega}^*(u)$ is *not* the standard measure boundary trace of u . In fact, when $\mu > 0$, the measure boundary trace of any \mathcal{L}_μ -harmonic function is zero.

In order to extend the definition of normalised boundary trace to arbitrary $\mu < 1/4$ we pick $\rho \in (0, \bar{\rho}]$ (with $\bar{\rho}$ as in Lemma 1.1) and employ (1.4) with $K_\mu^{\Omega_\rho}$ instead of K_μ^Ω . Since $C_H(\Omega_\rho) = 1/4$, $K_\mu^{\Omega_\rho}$ is well defined for every $\mu < 1/4$.

We show that if, for some ρ as above, there exists $\nu \in \mathfrak{M}_+(\partial\Omega)$ such that

$$(1.5) \quad \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^{\alpha_-}} \int_{\Sigma_\varepsilon} |u - \mathbb{K}_\mu^{\Omega_\rho}[\nu]| dS = 0$$

then (1.5) holds for every $\rho \in (0, \bar{\rho}]$ and the measure ν is independent of ρ .

In addition we show that a positive solution of equation (P_μ) possesses a normalised boundary trace if and only if it is a moderate solution.

1.4. Main results. We start with a few results about the linear operator.

Theorem 1.4. *Let $\mu < 1/4$. Suppose that u is positive and \mathcal{L}_μ -subharmonic in $\Omega_{\bar{\rho}}$. Then u has a normalized boundary trace on $\partial\Omega$ if and only if u is dominated in Ω_ρ (for some $\rho \in (0, \bar{\rho})$) by an \mathcal{L}_μ -harmonic function.*

Theorem 1.5. *Let $\mu < 1/4$. Suppose that u is a non-negative, \mathcal{L}_μ -subharmonic function in $\Omega_{\bar{\rho}}$. In addition assume that, for some $\rho \in (0, \bar{\rho})$ u is dominated in Ω_ρ by an \mathcal{L}_μ -harmonic function. Then, either*

(i) $\text{tr}_{\partial\Omega}^* u = 0$, in which case, for every $\beta \in (0, \rho)$ there exists a constant $c_\beta > 0$ such that

$$(1.6) \quad u(x) \leq c_\beta \delta(x)^{\alpha_+} \quad \text{in } \Omega_\beta$$

or

(ii) $\text{tr}_{\partial\Omega}^* u > 0$, in which case, for every β as above,

$$(1.7) \quad \frac{1}{c_\beta} \beta(x)^{\alpha_-} \leq \int_{\Sigma_\beta} u dS \leq c_\beta \beta^{\alpha_-} \quad \text{in } \Omega_\beta.$$

Theorem 1.6. *Let $\mu < 1/4$. Suppose that u is positive and \mathcal{L}_μ -superharmonic in $\Omega_{\bar{\rho}}$. If $\text{tr}_{\partial\Omega}^* u \neq 0$ then (1.7) holds.*

Corollary 1.7. *Suppose that u is non-negative and \mathcal{L}_μ -subharmonic in $\Omega_{\bar{\rho}}$. Then either (1.6) holds or*

$$(1.8) \quad 0 < \limsup_{\beta \rightarrow 0} \frac{1}{\beta^{\alpha_-}} \int_{\Sigma_\beta} u dS.$$

Remark 1.8. The corollary is an improved version of [5, Thm. 2.9]. Since we do not assume that u is dominated by an \mathcal{L}_μ -harmonic function the alternative to (1.6) is not necessarily (1.7) but only (1.8) which is nothing more than the negation of the statement $\text{tr}_{\partial\Omega}^* u = 0$.

Clearly every positive subsolution of the nonlinear equation (P_μ) is \mathcal{L}_μ -subharmonic so that the above results apply to it.

We turn to the nonlinear problem.

Theorem 1.9. *Let $\mu < 1/4$ and $\nu \in \mathfrak{M}^+(\partial\Omega) \setminus \{0\}$. Assume that $\mathbb{K}_\mu^{\Omega_\rho}[\nu] \in L_{\delta^{\alpha_+}}^q(\Omega_\rho)$ for some $\rho \in (0, \bar{\rho}]$. Then the boundary value problem (P_μ^ν) admits a positive solution u .*

We emphasise that if $C_H(\Omega) < 1/4$ then for $\mu \in [C_H(\Omega), 1/4)$ an \mathcal{L}_μ -harmonic extension of ν exists only locally in a strip Ω_ρ . Nevertheless, problem (P_μ^ν) has a positive solution in Ω , for any $\mu < 1/4$.

When $\mu < C_H(\Omega)$ problem (P_μ^ν) admits at most one solution for every $\nu \in \mathfrak{M}_+(\partial\Omega)$ [10]. However, if $C_H(\Omega) < \mu < 1/4$ then uniqueness fails. Indeed, it was proved in [5, Theorem 5.3] that in the latter case there exists a positive solution of (P_μ^ν) with $\nu = 0$. An alternative, more direct proof, of this result is presented in Appendix A.

Theorem 1.10. *Let u be a positive solution of (P_μ) . Then,*

- (i) *u has a normalized boundary trace if and only if $u \in L^q(\Omega; \delta^{\alpha_+})$.*
- (ii) *If u has normalized boundary trace ν then*

$$(1.9) \quad \lim_{x \rightarrow y} \frac{u(x)}{\mathbb{K}_\mu^{\Omega_\rho}[\nu](x)} = 1 \quad \text{non-tangentially, for } \nu\text{-a.e. } y \in \partial\Omega.$$

Theorem 1.11. *Let $\nu \in \mathfrak{M}_+(\partial\Omega)$. If $\mathbb{K}_\mu^{\Omega_\rho}[\nu] \in L^q(\Omega; \delta^{\alpha_+})$ then (P_μ^ν) has a solution.*

In general, the existence of a solution of (P_μ^ν) does not imply that $\mathbb{K}_\mu^{\Omega_\rho}[\nu] \in L^q(\Omega; \delta^{\alpha_+})$. In fact, for any $\mu > 0$ and $q > 1$, one can construct functions $f \in L^1(\partial\Omega)$ such that $\mathbb{K}_\mu^{\Omega_\rho}[f] \notin L^q(\Omega; \delta^{\alpha_+})$ while (P_μ^ν) has a solution whenever $\nu = f \in L^1(\partial\Omega)$.

Let

$$(1.10) \quad q_{\mu,c} := \frac{N + \alpha_+}{N - 1 - \alpha_-} \quad \forall \mu < 1/4.$$

The next result has been obtained in [10, Theorems E and F] for $\mu \in (0, C_H(\Omega))$. A similar result is presented in [8, Theorems D and E], under the assumption that Ω is a convex domain, in which case it is known that $C_H(\Omega) = 1/4$.

Proposition 1.12. *Let $\mu < 1/4$. If $1 < q < q_{\mu,c}$ then the boundary value problem (P_μ^ν) has a solution for every Borel measure $\nu \in \mathfrak{M}^+(\partial\Omega)$. Moreover, if $q \geq q_{\mu,c}$ then problem (P_μ^ν) has no solution when ν is the Dirac measure.*

In the next proposition, the existence statement is a consequence of Theorem 1.11. The non-existence part is more subtle.

Proposition 1.13. (i) For every $\mu < 1/4$ put

$$q_\mu^* = \begin{cases} \infty & \text{if } \mu \geq 0 \\ 1 - \frac{2}{\alpha_-} & \text{if } \mu < 0. \end{cases}$$

If $1 < q < q_\mu^*$ then problem (P_μ^ν) has a solution for every measure $\nu = f dS$, $f \in L^1(\partial\Omega)$.

(ii) If $q \geq q_\mu^*$ then problem (P_μ^ν) has no solution for any $\nu \in \mathfrak{M}_+(\partial\Omega) \setminus \{0\}$.

Remark 1.14. If $\mu < 0$ then $\alpha_- < 0$ so that $q_\mu^* > 1$ and $q_{\mu,c} < q_\mu^*$.

The paper is organised as follows. In Section 2 we study the linear problem. We derive estimates of the Green and Martin kernels of \mathcal{L}_μ in Ω_ρ and discuss the boundary behavior of local positive \mathcal{L}_μ -sub and superharmonic functions in terms of the normalized trace.

In Section 3 these results are applied to the study of the nonlinear boundary value problem (P_μ^ν) .

2. LINEAR EQUATION AND NORMALISED BOUNDARY TRACE

2.1. The local behavior of Green and Martin kernels. We recall some results concerning Schrödinger equations, that are needed in what follows. The results are due to Ancona [2]. Let D be a bounded Lipschitz domain and consider the Schrödinger operator $\mathcal{L}^V = \Delta + V$ where $V \in C(D)$ is a potential such that, for some constant $a > 0$, $|V(x)| \leq a \operatorname{dist}(x, \partial D)^{-2}$ and \mathcal{L}^V possesses a positive supersolution. (If $V \leq 0$ there is always a supersolution namely, $u = 1$.) Then \mathcal{L}^V has a Green function G^V and Martin kernel K^V in D . The Martin boundary coincides with ∂D and the following holds,

Theorem 2.1 (Representation Theorem). *For every $\nu \in \mathfrak{M}^+(\partial D)$ the function*

$$\mathbb{K}^V[\nu](x) := \int_{\partial D} K^V(x, y) d\nu(y), \quad x \in D,$$

is \mathcal{L}^V -harmonic in D . Conversely, if u is a positive \mathcal{L}^V -harmonic function in D then there exists a unique measure $\nu \in \mathfrak{M}^+(\partial D)$ such that $u = \mathbb{K}^V[\nu]$.

In order to state the boundary Harnack principle we need additional notation. Let $y \in \partial D$ and let $\xi = \xi^y$ be a local set of coordinates centered at y such that the ξ_1 -axis is in the direction of an interior pseudo normal \mathbf{n}_y . (If D is a C^1 domain we may take \mathbf{n}_y to be the interior unit normal.) Denote

$$T_y(r, \rho) = \{\xi = (\xi_1, \xi') : |\xi_1| < \rho, \quad |\xi'| < r\}.$$

Assume that r and ρ are so chosen that

$$\omega_y := T_y(r, \rho) \cap D = \{\xi : F_y(\xi') < \xi_1 < \rho, \quad |\xi'| < r\}$$

where F_y is a Lipschitz function in \mathbb{R}^{N-1} , with Lipschitz constant Λ , $F_y(0) = 0$ and $12\Lambda < \rho/r$. Since D is a bounded Lipschitz domain Λ, r, ρ can be chosen independently of $y \in \partial D$.

Let $A \in T(r, \rho)$ be the point such that $\xi(A) = (\rho/2, 0)$. Then the boundary Harnack principle reads as follows: If u, v are positive \mathcal{L}_μ -harmonic functions in ω_y vanishing continuously on $\partial\Omega \cap T_y(r, \rho)$ then

$$(2.1) \quad C^{-1} \frac{u(A)}{v(A)} \leq \frac{u(\xi)}{v(\xi)} \leq C \frac{u(A)}{v(A)} \quad \forall \xi \in T_y(r/2, \rho/2) \cap D,$$

where the constant C depends only on $N, M, \rho/r$ and the Lipschitz constant of F_y , say Λ . (Λ may be taken to be independent of $y \in \partial D$.)

We also need the following consequence of the boundary Harnack principle (c.f. Ancona [1, Lemma 3.5]): there exist positive numbers c, t_0 such that

$$(2.2) \quad c^{-1}|x - y|^{2-N} \leq K^V(x, y)G^V(x, x_0) \leq c|x - y|^{2-N}$$

for every $y \in \partial\Omega'$ and x on the interior pseudo normal at y such that $|x - y| \leq t_0$.

Recall that if $V(x) = \mu \text{dist}(x, \partial D)^{-2}$ and $\mu < C_H(D)$ then \mathcal{L}^V has a positive supersolution. In particular, if $D = \Omega_{\bar{\rho}}$ then $C_H(D) = 1/4$. Therefore, in this case, the above results apply to the operator $\mathcal{L}_\mu = \Delta + \frac{\mu}{\delta_\Omega^2}$ for every $\mu < 1/4$.

Notation. Let D be a subdomain of Ω and denote

$$\mathcal{L}_{\mu,D} = \Delta + \frac{\mu}{\delta_D^2} \quad \text{where} \quad \delta_D(x) = \text{dist}(x, \partial D).$$

Assume that $\mu < C_H(D)$ and let D' be a subdomain of D . Obviously $C_H(D') \geq C_H(D)$. Denote the Green kernel (resp. the Martin kernel) of \mathcal{L}_μ in D by G_μ^D (resp. K_μ^D). Denote the Green kernel (resp. the Martin kernel) of $\mathcal{L}_{\mu,D}$ in D' by $G_{\mu,D}^{D'}$ (resp. $K_{\mu,D}^{D'}$).

Lemma 2.2. *Assume that $\mu < 1/4$. Let $\bar{\rho}$ be as in Lemma 1.1 and $t \in (0, \bar{\rho})$. Put $U = \Omega_{\bar{\rho}} = [\delta(x) < \bar{\rho}]$, $\Omega_t = [\delta(x) < t]$, $U_t = [\bar{\rho} > \delta(x) > t]$. Then,*

$$(2.3) \quad \begin{aligned} G_\mu^{\Omega_{t/2}}(x, y) &\leq \\ C(t) \inf(|x - y|^{2-N}, \delta(x)^{\alpha_+} \delta(y)^{\alpha_+} |x - y|^{2\alpha_- - N}) &\quad \forall x, y \in \Omega_{t/2} \end{aligned}$$

Proof. Note that $\mathcal{L}_\mu = \mathcal{L}_{\mu,U}$ in $\Omega_{t/2}$. Hence

$$G_\mu^{\Omega_{t/2}} = G_{\mu,U}^{\Omega_{t/2}}.$$

It is well-known that the Green function is monotone with respect to the domain. Therefore $G_{\mu,U}^{\Omega_{t/2}} < G_{\mu,U}^{\Omega_t}$ which implies

$$(2.4) \quad G_\mu^{\Omega_{t/2}}(x, y) \leq c G_{\mu,U}^{\Omega_t}(x, y) \quad \forall x, y \in \Omega_{t/2}.$$

By (2.4) and the estimate of the Green function of $\mathcal{L}_{\mu,U}$ (see [7] and [10, (2.6)]),

$$(2.5) \quad \begin{aligned} G_\mu^{\Omega_{t/2}}(x, y) &\leq c G_{\mu,U}^{\Omega_t}(x, y) \leq c G_{\mu,U}^U(x, y) \\ &\sim \inf(|x - y|^{2-N}, \delta(x)^{\alpha_+} \delta(y)^{\alpha_+} |x - y|^{2\alpha_- - N}) \end{aligned}$$

for every $x, y \in \Omega_{t/2}$. This implies (2.3). \square

Theorem 2.3. Assume that $\mu < 1/4$, let $\bar{\rho}$ be as in Lemma 1.1 and let $t \in (0, \bar{\rho}/2)$. Using the notations of the previous lemma, pick $x_t \in U_t$ and $x'_t \in \Omega_t$ such that $\delta(x_t) = (t + \bar{\rho})/2$ and $\delta(x'_t) = t/2$. As usual G_0^U denotes the Green function for $-\Delta$ in U . A similar notation is employed for the corresponding Martin kernels. Then,

$$(2.6) \quad \begin{aligned} c_1(t)^{-1} G_{\mu,U}^U(x, x_t) &\leq G_\mu^U(x, x_t) \leq c_1(t) G_{\mu,U}^U(x, x_t) \quad \forall x \in \Omega_t \\ c_2(t)^{-1} G_0^U(x, x'_t) &\leq G_\mu^U(x, x'_t) \leq c_2(t) G_0^U(x, x'_t) \quad \forall x \in U_t, \end{aligned}$$

and

$$(2.7) \quad \begin{aligned} c_3(t)^{-1} K_{\mu,U}^U(x, y) &\leq K_\mu^U(x, y) \leq c_3(t) K_{\mu,U}^U(x, y) \quad \forall (x, y) \in \Omega_t \times \partial\Omega, \\ c_4(t)^{-1} K_0^U(x, y) &\leq K_\mu^U(x, y) \leq c_4(t) K_0^U(x, y) \quad \forall (x, y) \in U_t \times \Sigma_{\bar{\rho}}. \end{aligned}$$

Proof. Note that $\mathcal{L}_\mu = \mathcal{L}_{\mu,U}$ in $\Omega_{\bar{\rho}/2}$. Hence both $G_\mu^U(\cdot, x_t)$ and $G_{\mu,U}^U(\cdot, x_t)$ are \mathcal{L}_μ -harmonic in Ω_t and vanish on $\partial\Omega$. Therefore, by the boundary Harnack principle they are equivalent in a strip S along $\partial\Omega$. In addition they are continuous and bounded away from zero in $\Omega_t \setminus S$. This implies the first inequality in (2.6). For the second inequality: $G_\mu^U(\cdot, x'_t)$ is \mathcal{L}_μ -harmonic in U_t , $G_0^U(\cdot, x'_t)$ is Δ harmonic in U_t and $\mathcal{L}_\mu - \Delta = \mu/\delta(x)^2$ is bounded in U_t . Therefore, since they both vanish on $\Sigma_{\bar{\rho}}$, we can still apply the boundary Harnack principle (c.f. Ancona [4]) to deduce that they are equivalent in the strip U_t . This implies the second inequality in (2.6).

Recall that, $G_{\mu,U}^U(x, x_t) \sim \delta_U(x)^{\alpha+}$ in Ω_t for $t \in (0, \rho)$. (Of course the constants involved in this relation depend on t .) Since $\delta_\Omega \sim \delta_U$ in Ω_t , this fact and (2.6) imply,

$$(2.8) \quad G_\mu^U(x, x_t) \sim \delta_\Omega(x)^{\alpha+} \quad \forall x \in \Omega_t.$$

In what follows we use the notation introduced for the statement of the boundary Harnack principle. Let $y \in \partial\Omega$ and let $\xi = \xi_y$ be a local set of coordinates at y relative to U . Thus

$$\omega_y = T_y(r, \rho) \cap U = \{\xi : F_y(\xi') < \xi_1 < \rho, |\xi'| < r\}.$$

We assume that $\gamma = \rho/r > 12\Lambda$.

Since $K_\mu^U(\cdot, y)$ and $G_\mu^U(\cdot, x_t)$ satisfy the (classical) Harnack inequality (2.2) remains valid in $\mathcal{C}_y(b) \cap T_y(r, \rho)$. Therefore, assuming that $\rho < t < \bar{\rho}$,

$$(2.9) \quad K_\mu^U(\xi, y) G_\mu^U(\xi, x_t) \sim K_\mu^U((\xi_1, 0), y) G_\mu^U((\xi_1, 0), x_t) \sim |\xi|^{2-N}$$

for every $\xi \in \mathcal{C}_y(b) \cap T_y(r, \rho)$. By (2.8) and (2.9),

$$(2.10) \quad K_\mu^U(\xi, y) \sim |\xi|^{2-N} \delta(\xi)^{-\alpha+} \quad \forall \xi \in \mathcal{C}_y(b) \cap T_y(r, \rho).$$

Let η be a point in \mathbb{R}^{N-1} such that $0 < |\eta| < r/2$ and denote by P the point $(F_y(\eta), \eta)$ in the local coordinates ξ_y . Then $P \in \partial\Omega$ and $\xi_P := \xi_y - P$ is a standard set of local coordinates at P . Choose r_P, ρ_P such that $r_P = |\eta|/2$ and $\rho_P/r_P = \gamma$. Then,

$$|x - y| = |\xi_y| \sim |\xi'_y| \sim r_P \quad \forall x \in \Omega \cap T_P(r_P, \rho_P).$$

Let $A_P = (\rho_P/2, 0)$ in ξ_P coordinates, i.e., $A_P = (F_y(\eta) + \gamma r_P/2, \eta)$ in ξ_y coordinates. Pick b such that $\Lambda < b < 2\Lambda$. Then

$$F_y(\eta) + \rho_P/2 \geq -\Lambda|\eta| - \gamma r_P/2 = |\eta|(-\Lambda + \gamma/4) > 2\Lambda|\eta|.$$

Consequently, $F_y(\eta) < b|\eta| < F_y(\eta) + \rho_P/2$, which implies

$$A_P \in \mathcal{C}_y(b) := \{\xi_y = (\xi_1, \xi') : \xi_1 > b|\xi'|\}.$$

Observe that

$$\delta_\Omega(A_P) \sim \rho_P/2, \quad |\xi_y(A_P)| = |A_P - y| \sim (\rho_P^2 + r_P^2)^{1/2} \sim r_P.$$

Therefore, by (2.10),

$$K_\mu^U(A_P, y) \sim r_P^{2-N-\alpha+}.$$

In fact,

$$|x - y| = |\xi_y| \sim r_P \quad \forall x \in \Omega \cap T_P(r_P, \rho_P).$$

Therefore applying (2.1) in $\Omega \cap T_P(r_P, \rho_P)$ with $u(x) = K_\mu^U(x, y)$ we obtain,

$$\begin{aligned} (2.11) \quad K_\mu^U(x, y) &\sim K_\mu^U(A_P, y) \frac{G_\mu^U(x, x_t)}{G_\mu^U(A_P, x_t)} \sim r_P^{2-N-\alpha+} (\delta(x)/r_P)^{\alpha+} \\ &\sim |x - y|^{2-N-2\alpha+} \delta(x)^{\alpha+} = \delta(x)^{\alpha+} |x - y|^{2\alpha-N} \end{aligned}$$

for every $x \in \Omega \cap T_P(r_P/2, \rho_P/2)$. Combining (2.10) and (2.11), we obtain,

$$(2.12) \quad K_\mu^U(x, y) \sim |x - y|^{2-N-\alpha+} (\delta(x)/|x - y|)^{\alpha+} = \delta(x)^{\alpha+} |x - y|^{2\alpha-N}$$

for every $x \in T_y(r/2, \rho/2)$. As (2.12) holds uniformly with respect to $y \in \partial\Omega$ we conclude that there exists $r' > 0$ such that this relation holds for every $(x, y) \in \Omega_{r'} \times \partial\Omega$. Consequently, for every $t \in (0, \bar{\rho})$,

$$(2.13) \quad K_\mu^U(x, y) \sim |x - y|^{2-N-\alpha+} (\delta(x)/|x - y|)^{\alpha+} = \delta(x)^{\alpha+} |x - y|^{2\alpha-N}$$

for every $(x, y) \in \Omega_t \times \partial\Omega$ with similarity constants depending on t . Since $K_{\mu,U}^U$ behaves precisely in the same way (see [10, Sec. 2.2]) we obtain the first inequality in (2.7). The second inequality is proved in a similar way. \square

We state below two key results concerning the operator \mathcal{L}_μ in $U = \Omega_{\bar{\rho}}$. These have been recently proved in [10], with respect to the operator \mathcal{L}_μ in Ω under the assumption that $0 < \mu < C_H(\Omega)$. (In fact, the condition $\mu > 0$ is redundant and does not affect the proofs.) Since $C_H(\Omega_{\bar{\rho}}) = 1/4$, the results apply to the operator $\mathcal{L}_{\mu, \Omega_{\bar{\rho}}}$ for every $\mu < 1/4$. In view of the relation between the Martin kernels and Green functions of $\mathcal{L}_{\mu, \Omega_{\bar{\rho}}}$ and \mathcal{L}_μ in $\Omega_{\bar{\rho}}$, these results also apply to the operator \mathcal{L}_μ in $\Omega_{\bar{\rho}}$.

Theorem 2.4. (i) If $\nu_0 \in \mathfrak{M}^+(\partial\Omega) \setminus \{0\}$ then there exist positive numbers c and $\rho_0 < \bar{\rho}$ such that,

$$(2.14) \quad c^{-1} \|\nu_0\| \leq \frac{1}{\varepsilon^{\alpha-}} \int_{\Sigma_\varepsilon} \mathbb{K}_\mu^{\Omega_\rho}[\nu_0] dS \leq c \|\nu_0\|, \quad \varepsilon \in (0, \rho_0).$$

(ii) Let $\rho \in (0, \bar{\rho})$ and let τ be a Radon measure in $\Omega_{\bar{\rho}}$. Denote

$$\mathbb{G}_\mu^{\Omega_\rho}[\tau](x) := \int_{\Omega_\rho} G_\mu^{\Omega_\rho}(x, y) d\tau(y), \quad x \in \Omega_\rho.$$

If $\tau \in \mathfrak{M}_{\delta^{\alpha+}}^+(\Omega_\rho)$ then for every $0 < \varepsilon < \rho' < \rho$,

$$(2.15) \quad \frac{1}{\varepsilon^{\alpha-}} \int_{\Sigma_\varepsilon} \mathbb{G}_\mu^{\Omega_\rho}[\tau] dS_x \leq c \int_{\Omega_\rho} \delta^{\alpha+} d\tau,$$

where c is a constant depending on μ, ρ' , but not on ε . Moreover,

$$(2.16) \quad \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^{\alpha-}} \int_{\Sigma_\varepsilon} \mathbb{G}_\mu^{\Omega_\rho}[\tau] dS = 0.$$

Remark 2.5. If $\mathbb{G}_\mu^{\Omega_\rho}[\tau](x') < \infty$ for some point $x' \in \Omega_\rho$ then $\tau \in \mathfrak{M}_{\delta^{\alpha+}}^+(\Omega_\rho)$ and $\mathbb{G}_\mu^{\Omega_\rho}[\tau](x) < \infty$ for every $x \in \Omega_\rho$. This follows from the fact that there exists $c > 0$ such that for every fixed $x \in \Omega_\rho$,

$$\frac{1}{c} \delta(y)^{\alpha+} \leq G_\mu^{\Omega_\rho}(x, y) \leq c \delta(y)^{\alpha+} \quad \forall y \in \Omega_{\delta(x)/2}.$$

Proof. In view of (2.13), inequality (2.14) follows from [10, Corollary 2.11].

The proof of (2.15) and (2.16) is similar to that of [10, Proposition 2.12]. However several modifications are needed; therefore we provide the proof of these statements in detail.

We may assume that $\tau > 0$. Denote $v := \mathbb{G}_\mu^{\Omega_\rho}[\tau]$. We start with the proof of (2.15).

By Fubini's theorem and (2.6),

$$\begin{aligned} \int_{\Sigma_\beta} v dS_x &\leq c \left(\int_{\Omega} \int_{\Sigma_\beta \cap B_{\frac{\beta}{2}}(y)} |x - y|^{2-N} dS_x d\tau(y) \right. \\ &\quad \left. + \beta^{\alpha+} \int_{\Omega} \int_{\Sigma_\beta \setminus B_{\frac{\beta}{2}}(y)} |x - y|^{2\alpha- - N} dS_x \delta^{\alpha+}(y) d\tau(y) \right) = I_1(\beta) + I_2(\beta). \end{aligned}$$

Note that, if $x \in \Sigma_\beta$ and $|x - y| \leq \beta/2$ then $\beta/2 \leq \delta(y) \leq 3\beta/2$. Therefore

$$\begin{aligned} I_1(\beta) &\leq c_1 \beta^{-\alpha+} \int_{\Sigma_\beta \cap B_{\frac{\beta}{4}}(y)} |x - y|^{2-N} dS_x \int_{\Omega_\rho} \delta(y)^{\alpha+} d\tau(y) \\ &\leq c'_1 \beta^{1-\alpha+} \int_{\Omega_\rho} \delta(y)^{\alpha+} d\tau(y) = c'_1 \beta^{\alpha-} \int_{\Omega_\rho} \delta(y)^{\alpha+} d\tau(y) \end{aligned}$$

and

$$I_2(\beta) \leq c_2 \beta^{\alpha+} \int_{\beta/4}^{\infty} r^{2\alpha- - N} r^{N-2} dr \int_{\Omega_\rho} \delta(y)^{\alpha+} d\tau \leq c'_2 \beta^{\alpha-} \int_{\Omega_\rho} \delta(y)^{\alpha+} d\tau.$$

This implies (2.15).

Given $\ell \in (0, \|\tau\|_{\mathfrak{M}_{\delta^{\alpha+}}^\alpha(\Omega)})$ and $\beta_1 \in (0, \beta_0)$ put $\tau_1 = \tau \chi_{\bar{D}_{\beta_1}}$ and $\tau_2 = \tau - \tau_1$. Pick $\beta_1 = \beta_1(\ell)$ such that

$$(2.17) \quad \int_{\Omega_{\beta_1}} \delta(y)^{\alpha+} d\tau \leq \ell.$$

Thus the choice of β_1 depends on the rate at which $\int_{\Omega_\beta} \delta_+^\alpha d\tau$ tends to zero as $\beta \rightarrow 0$.

Put $v_i = \mathbb{G}_\mu^\Omega[\tau_i]$. Then, for $0 < \beta < \beta_1/2$,

$$\int_{\Sigma_\beta} v_1 dS_x \leq c_3 \beta^{\alpha_+} \beta_1^{2\alpha_- - N} \int_{\Omega_\rho} \delta^{\alpha_+}(y) d\tau_1(y).$$

Thus,

$$(2.18) \quad \lim_{\beta \rightarrow 0} \frac{1}{\beta^{\alpha_-}} \int_{\Sigma_\beta} v_1 dS_x = 0.$$

On the other hand, by (2.15) (replacing Ω_ρ by Ω_{β_1}) and (2.17),

$$(2.19) \quad \frac{1}{\beta^{\alpha_-}} \int_{\Sigma_\beta} v_2 dS_x \leq c\ell \quad \forall \beta < \beta_1.$$

This proves (2.16). \square

Corollary 2.6. *Let $\rho \in (0, \bar{\rho}]$ and assume that h is a nonnegative \mathcal{L}_μ -harmonic function in Ω_ρ such that*

$$(2.20) \quad \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^{\alpha_-}} \int_{\Sigma_\varepsilon} h dS = 0.$$

Then: (i) $h = \mathbb{K}_\mu^{\Omega_\rho}[\nu_\rho]$ for some measure $\nu_\rho \in \mathfrak{M}^+(\Sigma_\rho)$ and (ii) For $t \in (0, \bar{\rho})$,

$$(2.21) \quad h \sim \delta_\Omega^{\alpha_+} \quad \text{in } \Omega_t,$$

with the similarity constant depending on t .

Proof. (i) By the Representation Theorem, $h = \mathbb{K}_\mu^{\Omega_\rho}[\nu]$ for some $\nu \in \mathfrak{M}(\partial\Omega_\rho)$. By (2.14) and (2.20), $\nu_0 := \nu \mathbf{1}_{\partial\Omega} = 0$. Thus $\nu = \nu_\rho := \nu \mathbf{1}_{\Sigma_\rho}$.

(ii) This is a consequence of (i) and (2.13). \square

Corollary 2.7. *If $\tau \in \mathfrak{M}_{\delta^{\alpha_+}}^+(\Omega_\rho) \setminus \{0\}$ then there exists a positive constant $c = c(\tau)$ such that*

$$(2.22) \quad \mathbb{G}_\mu^{\Omega_\rho}[\tau](x) \geq c\delta(x)^{\alpha_+} \quad \forall x \in \Omega_\rho,$$

and

$$(2.23) \quad \liminf_{x \rightarrow \partial\Omega} \frac{\mathbb{G}_\mu^{\Omega_\rho}[\tau](x)}{\delta(x)^{\alpha_-}} < \infty.$$

Proof. Let $t \in (0, \rho)$ be a number such that $\tau(\Omega_\rho \setminus \Omega_t) > 0$. Let $\tau' \in \mathfrak{M}_+(\Omega_\rho)$ be defined by: $\tau' = \tau$ in $\Omega_\rho \setminus \Omega_t$ and $\tau' = 0$ in Ω_t . Then

$$\mathbb{G}_\mu^{\Omega_\rho}[\tau] \geq G_\mu^{\Omega_\rho}[\tau'] := h.$$

Since h is \mathcal{L}_μ -harmonic in Ω_t , (2.22) is a consequence of (2.21)

Inequality (2.23) follows from (2.15). \square

The next result was proved in [10] for \mathcal{L}_μ in a domain Ω such that $\mu < C_H(\Omega)$.

Theorem 2.8. *Let w be a nonnegative \mathcal{L}_μ -subharmonic function in Ω_ρ . If w is dominated by an \mathcal{L}_μ -superharmonic function in Ω_ρ then $\mathcal{L}_\mu w = \lambda \in \mathfrak{M}_{\delta^{\alpha_+}}^+(\Omega_\rho)$ and there exists $\nu \in \mathfrak{M}^+(\partial\Omega_\rho)$ such that*

$$(2.24) \quad w = \mathbb{K}_\mu^{\Omega_\rho}[\nu] - \mathbb{G}_\mu^{\Omega_\rho}[\lambda].$$

Proof. There exists a nonnegative Radon measure λ in Ω_ρ , such that $-\mathcal{L}_\mu w = -\lambda$ in Ω_ρ . Since w is dominated by an \mathcal{L}_μ -superharmonic function in Ω_ρ one shows, as in the proof of [10, Proposition 2.14], that $\lambda \in \mathfrak{M}_{\delta^{\alpha_+}}(\Omega_\rho)$. Then $v := w + \mathbb{G}_\mu^{\Omega_\rho}[\lambda]$ is a nonnegative \mathcal{L}_μ -harmonic function in Ω_ρ . By the Representation Theorem, $v = \mathbb{K}_\mu^{\Omega_\rho}[\nu]$ for some $\nu \in \mathfrak{M}^+(\partial\Omega_\rho)$. \square

Definition 2.9. A Borel function $u : \Omega \rightarrow \mathbb{R}$ possesses a *normalised boundary trace* $\nu_0 \in \mathfrak{M}^+(\partial\Omega)$ if, for some $\rho \in (0, \bar{\rho})$,

$$(2.25) \quad \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^{\alpha_-}} \int_{\Sigma_\varepsilon} |u - \mathbb{K}_\mu^{\Omega_\rho}[\nu_0]| dS = 0.$$

The normalised boundary trace on $\partial\Omega$ will be denoted by $\text{tr}_{\partial\Omega}^*(u)$.

Remark. Since u is a Borel function $u|_{\Sigma_\rho}$ is well defined and (2.25) implies that this function is in $L^1(\Sigma_\varepsilon)$ for all sufficiently small ε .

We say that u has a *measure boundary trace* on Σ_ρ if there exists $\nu_1 \in \mathfrak{M}^+(\Sigma_\rho)$ such that

$$\lim_{a \rightarrow \rho-0} \int_{\Sigma_a} u \phi dS \rightarrow \int_{\Sigma_\rho} \phi d\nu_1 \quad \forall \phi \in C_0(\bar{\Omega}_\rho).$$

This trace is denoted by $\text{tr}_{\Sigma_\rho}(u)$. If both $\text{tr}_{\Sigma_\rho}(u)$ and $\text{tr}_{\partial\Omega}^*(u)$ exist then the measure $\nu \in \mathfrak{M}_+(\partial\Omega_\rho)$ given by $\nu \mathbf{1}_{\partial\Omega} = \text{tr}_{\partial\Omega}^*(u)$ and $\nu \mathbf{1}_{\Sigma_\rho} = \text{tr}_{\Sigma_\rho}(u)$ is denoted by $\text{tr}_{\partial\Omega_\rho}^\mu(u)$.

Lemma 2.10. *The normalised boundary trace ν_0 is uniquely defined, independently of ρ .*

Proof. First we note that (2.25) remains valid if ν_0 is replaced by any measure $\nu \in \mathfrak{M}_+(\partial\Omega_\rho)$ such that $\nu_0 = \nu \mathbf{1}_{\partial\Omega}$. This follows from the fact that, for every measure $\nu_\rho \in \mathfrak{M}_+(\Sigma_\rho)$,

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^{\alpha_-}} \int_{\Sigma_\varepsilon} \mathbb{K}_\mu^{\Omega_\rho}[\nu_\rho] dS = 0.$$

This implies that if (2.25) holds with respect to some $\rho \in (0, \bar{\rho})$ then it is valid for any ρ' in this range. Suppose for instance that $\rho < \rho' < \bar{\rho}$ and put $v = \mathbb{K}_\mu^{\Omega_{\rho'}}[\nu_0]$. Let $\nu \in \mathfrak{M}_+(\partial\Omega_\rho)$ be the measure equal to ν_0 on $\partial\Omega$ and to $h = v|_{\Sigma_\rho} d\omega_\rho$ on Σ_ρ . (Here ω_ρ is the \mathcal{L}_μ -harmonic measure on Σ_ρ relative to $\Omega_{\rho'}$. Since Σ_ρ is ‘smooth’ ω_ρ is absolutely continuous with respect to surface measure.) Then $v = \mathbb{K}_\mu^{\Omega_\rho}[\nu]$ in Ω_ρ and

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^{\alpha_-}} \int_{\Sigma_\varepsilon} |\mathbb{K}_\mu^{\Omega_\rho}[\nu] - \mathbb{K}_\mu^{\Omega_\rho}[\nu_0]| dS = 0.$$

It remains to verify that, if (2.25) holds then ν_0 is uniquely determined by u in a fixed domain Ω_ρ .

Suppose, by negation, that there exist $\nu_1, \nu_2 \in \mathfrak{M}_+(\partial\Omega)$ such that (2.25) holds for both $v_1 = K_\mu^{\Omega_\rho}[\nu_1]$ and $v_2 = K_\mu^{\Omega_\rho}[\nu_2]$. Then $w := |v_1 - v_2|$ is \mathcal{L}_μ -subharmonic and $\text{tr}_{\partial\Omega}^*(w) = 0$.

Clearly w is dominated by the \mathcal{L}_μ -superharmonic function $v_1 + v_2$. Therefore, by Theorem 2.8 there exist $\lambda \in \mathfrak{M}_{\delta^{+}}^+(\Omega_\rho)$ and $\chi \in \mathfrak{M}^+(\partial\Omega_\rho)$ such that,

$$w = \mathbb{K}_\mu^{\Omega_\rho}[\chi] - \mathbb{G}_\mu^{\Omega_\rho}[\lambda].$$

Thus $w + \mathbb{G}_\mu^{\Omega_\rho}[\lambda]$ is \mathcal{L}_μ -harmonic. By (2.16) and the fact that $\text{tr}_{\partial\Omega}^* w = 0$ we have $\text{tr}_{\partial\Omega}^*(w + \mathbb{G}_\mu^{\Omega_\rho}[\lambda]) = 0$. Hence $w = 0$ and therefore $\nu_1 = \nu_2$. \square

Theorem 2.11. *Let w be a nonnegative \mathcal{L}_μ -subharmonic function in Ω_ρ dominated by an \mathcal{L}_μ -superharmonic function in this domain. Then the boundary trace $\nu = \text{tr}_{\partial\Omega_\rho}^\mu(w)$ is well-defined and*

$$(2.26) \quad w \leq \mathbb{K}_\mu^{\Omega_\rho}[\nu].$$

If $\nu_0 := \nu \mathbf{1}_{\partial\Omega}$ then,

$$(2.27) \quad \lim_{x \rightarrow \partial\Omega} \frac{w(x)}{\mathbb{K}_\mu^{\Omega_\rho}[\nu_0](x)} = 1 \quad \text{non-tangentially, } \nu_0\text{-a.e. on } \partial\Omega.$$

If $\nu_0 = 0$ then,

$$(2.28) \quad \limsup_{x \rightarrow \partial\Omega} \frac{w(x)}{\delta^{\alpha+}(x)} < \infty.$$

Proof. The first statement (2.26) follows from (2.24) and Theorem 2.4 (ii).

The second statement (2.27) follows from (2.24) and the fact that $\mathbb{G}_\mu^{\Omega_\rho}[\lambda]$ is an \mathcal{L}_μ -potential (i.e. a positive superharmonic function that does not dominate any positive \mathcal{L}_μ -harmonic function). This fact implies (see, e.g. [3]):

$$\lim_{x \rightarrow \partial\Omega} \frac{\mathbb{G}_\mu^{\Omega_\rho}[\lambda](x)}{\mathbb{K}_\mu^{\Omega_\rho}[\nu](x)} \rightarrow 0 \quad \nu\text{-a.e. on } \partial\Omega.$$

By Fatou's limit theorem

$$\lim_{x \rightarrow \partial\Omega} \frac{\mathbb{K}_\mu^{\Omega_\rho}[\nu_0](x)}{\mathbb{K}_\mu^{\Omega_\rho}[\nu](x)} = 1 \quad \nu\text{-a.e. on } \partial\Omega.$$

Therefore (2.24) implies (2.27).

The third statement (2.28) follows from (2.26) and Corollary 2.6. \square

Corollary 2.12. *Let w be a nonnegative \mathcal{L}_μ -subharmonic function in Ω_ρ for some $\rho \in (0, \bar{\rho})$. Then w possesses a normalised boundary trace in $\mathfrak{M}^+(\partial\Omega)$ if and only if w is dominated by a positive \mathcal{L}_μ -superharmonic function v in a strip around $\partial\Omega$.*

Proof. If w is dominated by a positive \mathcal{L}_μ -superharmonic function in Ω_ρ then the existence of $\text{tr}_{\partial\Omega}^*(w)$ follows from (2.16) and Theorem 2.8.

Next suppose that w has a normalized boundary trace $\nu_0 \in \mathfrak{M}^+(\partial\Omega)$. Without loss of generality we may assume that it also has a measure boundary trace ν_ρ on Σ_ρ . Since u is \mathcal{L}_μ -subharmonic, there exists a positive Radon measure τ in Ω such that

$$-\mathcal{L}_\mu u = -\tau.$$

Let $\tau_\beta := \tau \mathbf{1}_{D_\beta \setminus \bar{D}_\rho}$, $w = \mathbb{K}_\mu^{\Omega_\rho}[\nu_0 + \nu_\rho]$ and $\nu_\beta = w|_{\Sigma_\beta}$.

Let u_β be the solution of the boundary value problem,

$$\begin{aligned} -\mathcal{L}_\mu v &= -\tau_\beta \text{ in } D_\beta \setminus \bar{D}_\rho, \\ v &= \nu_\rho \text{ on } \Sigma_\rho, \quad v = \nu_\beta \text{ on } \Sigma_\beta. \end{aligned}$$

Then

$$u_\beta + \mathbb{G}_\mu^{D_\beta \setminus \bar{D}_\rho}[\tau_\beta] = w.$$

It follows that

$$G_\mu^{\Omega_\rho}[\tau] = \lim_{\beta \rightarrow 0} \mathbb{G}_\mu^{D_\beta \setminus \bar{D}_\rho}[\tau_\beta] < \infty,$$

which in turn implies that $\tau \in \mathfrak{M}_+(\Omega; \delta^{\alpha+})$ and finally

$$u + G_\mu^{\Omega_\rho}[\tau] = w.$$

In particular,

$$(2.29) \quad u \leq w = \mathbb{K}_\mu^{\Omega_\rho}[\nu_0 + \nu_\rho].$$

□

Corollary 2.13. (i) Suppose that u is positive and \mathcal{L}_μ -subharmonic in $\Omega_{\bar{\rho}}$. Then $\text{tr}_{\partial\Omega}^* u = 0$ if and only if, for every $\rho \in (0, \bar{\rho})$, there exists a constant c_ρ such that

$$(2.30) \quad u(x) \leq c_\rho \delta(x)^{\alpha+} \quad \forall x \in \Omega_\rho.$$

(ii) Suppose that u is positive and \mathcal{L}_μ -superharmonic in $\Omega_{\bar{\rho}}$. Then u has a normalized boundary trace $\nu \in \mathfrak{M}_+(\partial\Omega)$ and consequently there exists c_ρ such that

$$(2.31) \quad \int_{\Sigma_\beta} u dS \leq c_\rho \beta^{\alpha-} \quad \forall \beta \in (0, \rho).$$

Proof. (i) Obviously (2.30) implies that $\text{tr}_{\partial\Omega}^*(u) = 0$. Conversely assume that $\text{tr}_{\partial\Omega}^*(u) = 0$.

By the previous corollary u is dominated by an \mathcal{L}_μ -harmonic function. Therefore, by Theorem 2.8, there exist $\lambda \in \mathfrak{M}_{\delta^{\alpha+}}^+(\Omega_\rho)$ and $\nu \in \mathfrak{M}^+(\partial\Omega_\rho)$ such that $u = \mathbb{K}_\mu^{\Omega_\rho}[\nu] - \mathbb{G}_\mu^{\Omega_\rho}[\lambda]$. Since $\text{tr}_{\partial\Omega}^*(u) = 0$, $\nu_0 = \nu \mathbf{1}_{\partial\Omega} = 0$. Hence $u < \mathbb{K}_\mu^{\Omega_\rho}[\nu_\rho]$ where $\nu_\rho = \nu \mathbf{1}_{\Sigma_\rho}$. Therefore the result follows from Corollary 2.6.

(ii) By the Riesz decomposition theorem (see [3]), $u = u_p + u_h$ where u_p is an \mathcal{L}_μ -potential and u_h is a nonnegative \mathcal{L}_μ -harmonic function in Ω_ρ . It is known that every \mathcal{L}_μ -potential is the Green potential of a positive measure. Thus there exists $\tau \in \mathfrak{M}_+(\Omega; \delta^{\alpha+})$ such that $u_p = \mathbb{G}_\mu^{\Omega_\rho}[\tau]$. By the Representation Theorem $u_h = \mathbb{K}_\mu^{\Omega_\rho}[\nu]$ for some $\nu \in \mathfrak{M}_+(\partial\Omega_\rho)$. Thus

$$(2.32) \quad u = \mathbb{G}_\mu^{\Omega_\rho}[\tau] + \mathbb{K}_\mu^{\Omega_\rho}[\nu].$$

The required result follows from Theorem 2.4. \square

3. \mathcal{L}_μ -MODERATE SOLUTIONS OF NONLINEAR EQUATION

In this section we study the nonlinear equation

$$(P_\mu) \quad -\mathcal{L}_\mu u + |u|^{q-1}u = 0 \quad \text{in } \Omega,$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $\mu < 1/4$ and $q > 1$.

3.1. Preliminaries. Suppose that $u \in L_{\text{loc}}^q(\Omega)$ is either a subsolution or a supersolution of (P_μ) , in the distributional sense. Then, $u \in W_{\text{loc}}^{1,p}(\Omega)$ for $1 \leq p < N/(N-1)$. If, in addition, u is a distributional *solution* of (P_μ) then it is also a classical solution.

Consequently, if $u \in L_{\text{loc}}^q(\Omega)$ is a distributional subsolution in Ω then

$$(3.1) \quad \int_{\Omega} \nabla u \cdot \nabla \varphi \, dx - \int_{\Omega} \frac{\mu}{\delta^2} u \varphi \, dx + \int_{\Omega} |u|^{q-1} u \varphi \, dx \leq 0 \quad \forall 0 \leq \varphi \in C_c^\infty(\Omega).$$

If, in addition, $u \in H_{\text{loc}}^1(\Omega)$ then (3.1) holds for every $\varphi \in H_c^1(\Omega)$.

A similar statement holds for supersolutions, in which case the inequality sign in (3.1) is inversed. Of course these statements remain valid for local subsolutions and supersolutions (in a subdomain $G \subset \Omega$).

We state below two results from [5] that will be used in the sequel.

Lemma 3.1. (COMPARISON PRINCIPLE [5, Lemma 3.2])

- (i) Let G be open with $\overline{G} \subset \Omega$. Let $0 \leq \underline{u}, \overline{u} \in H_{\text{loc}}^1(G) \cap C(\overline{G})$ be a pair of sub and supersolutions to (P_μ) in G such that

$$\limsup_{x \rightarrow \partial G} [\underline{u}(x) - \overline{u}(x)] < 0.$$

Then $\underline{u} \leq \overline{u}$ in G .

- (ii) Let G be open with $\overline{G} \subset \Omega$. Let $\underline{u}, \overline{u} \in H^1(G) \cap C(\overline{G})$ be a pair of sub and supersolutions to (P_μ) in G and $\underline{u} \leq \overline{u}$ on ∂G . Then $\underline{u} \leq \overline{u}$ in G .

Lemma 3.2. ([5, Lemma 4.10]) Assume that (P_μ) admits a subsolution \underline{u} and a supersolution \overline{u} in Ω so that $0 \leq \underline{u} \leq \overline{u}$ in Ω . Then (P_μ) has a solution U in Ω such that $\underline{u} \leq U \leq \overline{u}$ in Ω .

In [5, Proposition 3.5] the Keller–Osserman estimate has been extended to equation (P_μ) . Specifically it was proved that every subsolution u of (P_μ) in Ω satisfies,

$$(3.2) \quad u(x) \leq \gamma_* \delta^{-\frac{2}{q-1}}(x) \quad \text{in } \Omega,$$

where γ_* is a constant independent of u . In addition it was shown that, if u is a local subsolution in Ω_ρ , continuous at Σ_ρ , then u satisfies (3.2) in Ω_ρ , but γ_* may depend on u . We prove below a stronger version that is used later on.

Lemma 3.3. (KELLER–OSSERMAN ESTIMATE) If u is a subsolution of (P_μ) in Ω then it satisfies (3.2) with a constant depending only on q, N, μ . If u is a subsolution of (P_μ) in Ω_ρ then (3.2) holds with a constant depending only on q, N, μ, ρ and $\delta(x)$ replaced by $\delta_\rho(x) := \text{dist}(x, \partial\Omega_\rho)$.

Proof. Without loss of generality we may assume that $u \geq 0$ because u_+ is a subsolution. If $\mu \leq 0$ then u is also a subsolution of the equation $-\Delta u + u^q = 0$. Therefore in this case (3.2) is a direct consequence of the classical Keller–Osserman inequality.

Now assume that $\mu > 0$. Let $y \in \Omega$ and $R = \delta(y)/2$. Then,

$$-\Delta u - \frac{\mu}{R^2}u + u^q \leq 0 \quad \text{in } B_R(y).$$

Therefore in $B_R(y)$ either $u \leq (8\mu/R^2)^{\frac{1}{q-1}}$ or $-\Delta u + u^q/2 \leq 0$. Hence, by Kato's inequality, the function $v := (u - (8\mu/R^2)^{\frac{1}{q-1}})_+$ satisfies

$$-\Delta v + v^q/2 \leq 0 \quad \text{in } B_R(y).$$

By the classical Keller–Osserman inequality,

$$v(y) \leq c(q, N)R^{-\frac{2}{q-1}}.$$

Since $u(y) \leq v(y) + (8\mu/R^2)^{\frac{1}{q-1}}$ we conclude that

$$(3.3) \quad u(y) \leq c(\mu, q, N)\delta_\Omega(y)^{-\frac{2}{q-1}} \quad \forall y \in \Omega.$$

Next, let u be a subsolution in Ω_ρ . As before we may assume that $u \geq 0$ and that $\mu > 0$. By the first part of the proof, (3.3) holds in $\Omega_{3\rho/4}$. Further,

$$-\Delta u - (4\mu/\rho^2)u + u^q \leq 0 \quad \text{in } \Omega'_\rho = \{x : \rho/2 \leq \delta(x) < \rho\}.$$

Therefore, either $u \leq (8\mu/\rho^2)^{\frac{1}{q-1}}$ or $-\Delta u + u^q/2 \leq 0$. By the same argument as before, the function $v := (u - (8\mu/\rho^2)^{\frac{1}{q-1}})_+$ satisfies

$$v(x) \leq c(q, N)\text{dist}(x, \Sigma_\rho)^{-\frac{2}{q-1}} \quad \forall x : 3\rho/4 \leq \delta(x) < \rho.$$

Consequently,

$$(3.4) \quad u(x) \leq c(\mu, q, N, \rho)\text{dist}(x, \partial\Omega_\rho)^{-\frac{2}{q-1}} \quad \forall x \in \Omega_\rho. \quad \square$$

3.2. Moderate solutions. We study the generalised boundary trace problem (P_μ^ν) where $\mu < 1/4$, $q > 1$ and $\nu \in \mathcal{M}^+(\partial\Omega)$. First we prove,

Lemma 3.4. *Let D be a C^2 domain such that $D \Subset \Omega$. If $0 \leq f \in C(\partial D)$ then there exists a unique solution of the problem*

$$(3.5) \quad \begin{cases} -\mathcal{L}_\mu u + u^q = 0 & \text{in } D, \\ u = f & \text{on } \partial D. \end{cases}$$

Proof. For $u \in H^1(D)$, let

$$J_D(u) = \int_D \left(\frac{1}{2}|\nabla u|^2 - \frac{\mu}{2\delta_\Omega^2}u^2 + \frac{1}{q+1}|u|^{q+1} \right) dx.$$

Since $\mu\delta_\Omega^{-2} \in L^\infty(D)$, it is standard to see that J_D is coercive and weakly l.s.c. on

$$H_f^1(D) = \{u \in H^1(D) : u = f \text{ on } \partial D\}.$$

Therefore there exists a minimizer $u_f \in H_f^1(D)$. We may assume that $u_f > 0$ because $|u_f|$ too is a minimizer. The minimizer is a solution of (3.5). The uniqueness is a consequence of the comparison principle. \square

Next consider the problem,

$$(P_\mu^\nu(\rho)) \quad \begin{cases} -\mathcal{L}_\mu u + u^q &= 0 & \text{in } \Omega_\rho, \\ \text{tr}_{\partial\Omega}^*(u) &= \nu \mathbf{1}_{\partial\Omega} =: \nu_0, \\ \text{tr}_{\Sigma_\rho}(u) &= \nu \mathbf{1}_{\Sigma_\rho} =: \nu_\rho. \end{cases}$$

where $\mu < 1/4$, $q > 1$, $\nu \in \mathcal{M}^+(\partial\Omega_\rho)$ and $\rho \in (0, \bar{\rho}]$.

The following result is an adaptation of [10, Theorem C] to problem $(P_\mu^\nu(\rho))$. Since $C_H(\Omega_{\bar{\rho}}) = 1/4$ the result applies to every $\mu < 1/4$. The proof follows the argument in [10]; for the convenience of the reader it is presented below.

Proposition 3.5. *Let $\nu \in \mathfrak{M}^+(\partial\Omega_\rho)$ and assume that $\mathbb{K}_\mu^{\Omega_\rho}[\nu] \in L_{\delta^{\alpha_+}}^q(\Omega_\rho)$ for some $\rho \in (0, \bar{\rho}]$. Then $(P_\mu^\nu(\rho))$ admits a unique solution U_ν .*

Proof. Let $\{D_n\}$ be a sequence of C^2 domains such that $\bar{D}_n \subset D_{n+1}$ and $D_n \uparrow \Omega_\rho$. Let u_n be the solution of (3.5) with $D = D_n$ and $f = f_n := \mathbb{K}_\mu^{\Omega_\rho}[\nu]|_{\partial D_n}$. Since $\mathbb{K}_\mu^{\Omega_\rho}[\nu]$ is a supersolution of the equation $\mathcal{L}_\mu v + v^q = 0$ in Ω_ρ it follows that u_n decreases and $u = \lim u_n$ is a solution of this equation. We claim that u is a solution of $(P_\mu^\nu(\rho))$. Indeed,

$$(3.6) \quad u_n + \mathbb{G}_\mu^{D_n}[u_n^q] = \mathbb{P}_\mu^{D_n}[f_n] = \mathbb{K}_\mu^{\Omega_\rho}[\nu] \quad \text{in } D_n,$$

where $\mathbb{P}_\mu^{D_n}$ denotes the Poisson kernel of \mathcal{L}_μ in D_n .

Since $u_n \leq \mathbb{K}_\mu^{\Omega_\rho}[\nu] \in L_{\delta^{\alpha_+}}^q(\Omega)$ it follows that

$$\mathbb{G}_\mu^{D_n}[u_n^q] \rightarrow \mathbb{G}_\mu^{\Omega_\rho}[u^q].$$

Hence, by (3.6),

$$u + \mathbb{G}_\mu^{\Omega_\rho}[u^q] = \mathbb{K}_\mu^{\Omega_\rho}[\nu] \quad \text{in } \Omega_\rho.$$

By Theorem 2.4, $\text{tr}_{\partial\Omega}^*(u) = \nu \mathbf{1}_{\partial\Omega}$ and (by (2.7)) $\text{tr}_{\Sigma_\rho}(u) = \nu \mathbf{1}_{\Sigma_\rho}$. \square

The next result is an adaptation of [10, Theorem D]. We omit the proof which – except for obvious modifications – is the same as in [10].

Proposition 3.6. *Assume that u is a positive solution of $(P_\mu^\nu(\rho))$. Then*

$$(3.7) \quad \lim_{x \rightarrow \partial\Omega} \frac{u(x)}{\mathbb{K}_\mu^{\Omega_\rho}[\nu_0](x)} = 1 \quad \text{non-tangentially, } \nu\text{-a.e. on } \partial\Omega,$$

where $\nu_0 = \nu \mathbf{1}_{\partial\Omega}$.

Theorem 3.7. *Let $\nu \in \mathfrak{M}^+(\partial\Omega)$ and $\rho \in (0, \bar{\rho})$. Let $\nu' \in \mathfrak{M}^+(\partial\Omega_\rho)$ be defined by $\nu' = \nu$ on $\partial\Omega$ and $\nu' = 0$ on Σ_ρ . Assume that, for some ρ as above, $\mathbb{K}_\mu^{\Omega_\rho}[\nu'] \in L_{\delta^{\alpha_+}}^q(\Omega_\rho)$. Then the boundary value problem $(P_\mu^{\nu'})$ admits a solution in Ω .*

Proof. By Proposition 3.5 there exists a (unique) solution $U_{\nu,0}$ of problem $(P_\mu^{\nu'}(\rho))$. For every $k \geq 0$, let $\nu_k \in \mathfrak{M}^+(\partial\Omega_\rho)$ be the measure given by, $\nu_k \mathbf{1}_{\partial\Omega} = \nu$ and $\nu_k \mathbf{1}_{\Sigma_\rho} = k dS_{\Sigma_\rho}$. By the same proposition there exists a (unique) solution $U_{\nu,k}$ of $(P_\mu^{\nu_k}(\rho))$. Put

$$U_{\nu,\infty} = \lim_{k \rightarrow \infty} U_{\nu,k}.$$

Let $R \in (0, \rho)$. By Lemma 3.4 there exists a unique solution v_R of (3.5) in D_R with $f = U_{\nu,0}|_{\Sigma_R}$. By the comparison principle,

$$U_{\nu,0} \leq v_R \leq U_{\nu,\infty} \quad \text{in } \Omega_\rho \cap D_R.$$

By Proposition 3.3 the family $\{v_R : 0 < R < \rho\}$ is bounded in compact subsets of Ω . Therefore there exists a sequence $\{R_j\}$ converging to zero such that v_{R_j} converges to a solution v of the nonlinear equation in Ω . By construction,

$$U_{\nu,0} \leq v \leq U_{\nu,\infty} \quad \text{in } \Omega_\rho.$$

Therefore $\text{tr}_{\partial\Omega}^*(v) = \nu$. □

Remark 3.8. If $\mu < C_H(\Omega)$ then problem (P_μ^ν) has at most one solution, [10, Theorem B]. However uniqueness fails when $C_H(\Omega) < \mu < 1/4$. It was proved in [5, Theorem 5.3] that in this case there exists a positive solution of (P_μ^ν) with $\nu = 0$. An alternative, more direct proof, is presented in Appendix A.

Proposition 3.9. *Assume that $u \in L_{loc}^q(\Omega)$ is a positive solution of (P_μ) . Then the following assertions are equivalent:*

- (i) u has a normalized boundary trace,
- (ii) u is a moderate solution in the sense of Definition 1.3,
- (iii) $u \in L^q(\Omega; \delta^{\alpha+})$.

Proof. The assumption implies that $\mathcal{L}_\mu u \leq 0$ in Ω . If $\rho \in (0, \bar{\rho}]$ then, by Lemma 2.12, (i) holds if and only if u is dominated by an \mathcal{L}_μ -superharmonic function in Ω_ρ . Consequently, by Lemma 3.2, (i) holds if and only if u is dominated by an \mathcal{L}_μ -harmonic function in Ω_ρ . Thus (i) and (ii) are equivalent.

If (iii) holds then $v := u + \mathbb{G}_\mu^{\Omega_\rho}[u^q]$ is \mathcal{L}_μ -harmonic. By the representation theorem there exists $\nu \in \mathfrak{M}(\partial\Omega_\rho)$ such that $v = \mathbb{K}_\mu^{\Omega_\rho}[\nu]$. Since $\text{tr}_{\partial\Omega}^* \mathbb{G}_\mu^{\Omega_\rho}[u^q] = 0$ it follows that $\nu \mathbf{1}_{\partial\Omega}$ is the normalized boundary trace of u . Conversely if (ii) holds then by Theorem 2.8 $\mathcal{L}_\mu u = u^q \in \mathfrak{M}_{\delta^{\alpha+}}^+(\Omega_\rho)$ which is the same as (iii). □

3.3. Critical exponents. The next result provides necessary and sufficient conditions in order that a positive measures $\nu \in \mathfrak{M}^+(\partial\Omega)$ satisfy,

$$(3.8) \quad \mathbb{K}_\mu^{\Omega_\rho}[\nu] \in L_{\delta^{\alpha+}}^q(\Omega_\rho) \quad \text{for some } \rho > 0.$$

Let $\Gamma_a(x-y) = |x-y|^{-(N-a)}$ denote the Riesz kernel of order $0 < a < N$ in \mathbb{R}^N .

Proposition 3.10. *Let $\nu \in \mathcal{M}^+(\partial\Omega)$.*

- (i) *If $\Gamma_1 * \nu \in L_{\delta^{1+(q-1)\alpha_-}}^q(\Omega)$ then ν satisfies (3.8).*
 - (ii) *Assume $\mu \geq 0$. If ν satisfies (3.8) then $\mathbb{P}_0^\Omega[\nu] \in L_{\delta^{1+(q-1)\alpha_-}}^q(\Omega)$.*
- Here P_0^Ω is the Poisson kernel of $-\Delta$ in Ω : $P_0^\Omega(x, y) = \delta(x)|x-y|^{-N}$.*

Proof. By (2.13),

$$(3.9) \quad \begin{aligned} K_\mu^{\Omega_\rho}(x, y) &\sim \frac{\delta(x)^{\alpha_+}}{|x - y|^{N-2\alpha_-}} \sim \delta(x)^{\alpha_-} P_0^\Omega(x, y) (|x - y|/\delta(x))^{2\alpha_-} \\ &\sim \delta(x)^{\alpha_-} \Gamma_1(x - y) (|x - y|/\delta(x))^{-1+2\alpha_-}, \end{aligned}$$

for every $(x, y) \in \Omega_{\rho/2} \times \partial\Omega$.

For every $\mu < 1/4$ we have $-1 + 2\alpha_- < 0$. Consequently,

$$(3.10) \quad K_\mu^{\Omega_\rho}(x, y) \leq c\delta(x)^{\alpha_-} \Gamma_1(x - y) \quad \forall (x, y) \in \Omega_{\rho/2} \times \partial\Omega.$$

Hence,

$$\|\mathbb{K}_\mu^{\Omega_\rho} \nu\|_{L_{\delta^{\alpha_+}}^q(\Omega_{\rho/2})}^q \leq c \int_{\Omega_{\rho/2}} \left(\int_{\partial\Omega} \Gamma_1(x - y) d\nu(y) \right)^q \delta(x)^{q\alpha_- + \alpha_+} dx.$$

This proves (i).

If $\mu \geq 0$, so that $\alpha_- \geq 0$ then, by (3.9),

$$(3.11) \quad K_\mu^{\Omega_\rho}(x, y) \geq c\delta(x)^{\alpha_-} P_0^\Omega(x, y) \quad \forall (x, y) \in \Omega_{\rho/2} \times \partial\Omega.$$

Therefore

$$\|\mathbb{K}_\mu^{\Omega_\rho} [\nu]\|_{L_{\delta^{\alpha_+}}^q(\Omega_{\rho/2})}^q \geq c \int_{\Omega_{\rho/2}} \left(\int_{\partial\Omega} P_0^\Omega(x, y) d\nu(y) \right)^q \delta(x)^{q\alpha_- + \alpha_+} dx.$$

This proves (ii). \square

Using this result we provide a necessary and sufficient condition for the existence of positive moderate solutions of (P_μ) .

Proposition 3.11. *Let $\nu \in \mathcal{M}^+(\partial\Omega)$.*

(i) *If $\alpha_- > -\frac{2}{q-1}$ then the boundary value problem (P_μ^ν) has a solution for every measure $\nu = f dS_{\partial\Omega}$ such that $f \in L^1(\partial\Omega)$.*

(ii) *If $\alpha_- \leq -\frac{2}{q-1}$ then, for every $\nu \gneq 0$, (P_μ^ν) has no solution.*

Remark. When $\mu > 0$ and consequently $\alpha_- > 0$, the condition in (i) holds for every $q > 1$.

Proof. Let $\nu = f dS_{\partial\Omega}$ and $f \in L^\infty(\partial\Omega)^+$. Let $x \in \Omega_{\beta_0}$ and pick $x' \in \partial\Omega$ such that $|x' - x| = \delta(x)$. Then,

$$(3.12) \quad \begin{aligned} \int_{\partial\Omega} |x - y|^{1-N} f(y) dS(y) &\leq c\|f\|_{L^\infty} \left(\int_{\substack{y \in \partial\Omega \\ |x' - y| \geq \delta(x)}} |x' - y|^{1-N} dS(y) + 1 \right) \\ &\leq c\|f\|_{L^\infty} (1 + |\ln \delta(x)|) \leq c'\|f\|_{L^\infty} |\ln \delta(x)|, \end{aligned}$$

where c' is independent of x . Therefore, if $(q-1)\alpha_- + 1 > -1$ then $\Gamma_1 * \nu \in L_{\delta^{1+(q-1)\alpha_-}}^q(\Omega)$. Consequently, by Proposition 3.10 (i) and Theorem 3.7, problem (P_μ^ν) has a solution.

Next, let $f \in L^1(\partial\Omega)^+$ and $\nu = f dS_{\partial\Omega}$. If $\nu_n = \min(f, n) dS_{\partial\Omega}$ then problem $(P_\mu^{\nu_n})$ has a solution u_n and the sequence $\{u_n\}$ is non-decreasing. In view of the Keller–Osserman estimate (3.2), $\{u_n\}$ converges to a solution u of (P_μ^ν) . This proves (i).

We turn to part (ii). Suppose that $\alpha_- \leq -\frac{2}{q-1}$ and that there exists $\nu \in \mathcal{M}^+(\partial\Omega) \setminus \{0\}$ such that problem (P_μ^ν) has a solution u . Then, there exists $c > 0$ such that

$$c\beta^{-\frac{2}{q-1}} \leq c\beta^{\alpha_-} \leq \int_{\Sigma_\beta} \mathbb{K}_\mu^{\Omega_\rho}[\nu] dS \quad \forall \beta \in (0, \beta_0).$$

Since $u = -\mathbb{G}_\mu[u^q] + \mathbb{K}_\mu[\nu]$ and $\text{tr}_{\partial\Omega}^*(\mathbb{G}_\mu[u^q]) = 0$ it follows that, for sufficiently small β_1 ,

$$(3.13) \quad c\beta^{\alpha_-} \leq \int_{\Sigma_\beta} u dS \quad \forall \beta \in (0, \beta_1).$$

But, by the Keller-Osserman estimate, $u(x) \leq c_1 \delta(x)^{-\frac{2}{q-1}}$ so that

$$(3.14) \quad c\beta^{\alpha_-} \leq \int_{\Sigma_\beta} u dS \leq c_2 \beta^{-\frac{2}{q-1}} \quad \forall \beta \in (0, \beta_1).$$

If $\alpha_- < -2/(q-1)$ we reached a contradiction. If $\alpha_- = -2/(q-1)$ then, in view of the Keller-Osserman estimate (3.2) we conclude that $u(x) \sim \delta(x)^{-\frac{2}{q-1}}$. This implies that $u \sim U_{\max}$ (= the maximal solution of $-\mathcal{L}_\mu v + v^q = 0$). Thus $\sup U_{\max}/u := c < \infty$. Now cu is a supersolution and, if v is the largest solution dominated by cu then $\text{tr}^*(v) = c \text{tr}^*(u) = c\nu$. It follows that $U_{\max} \leq v$ which is impossible. \square

Remark 3.12. When $\mu > 0$ – and consequently $\alpha_- > 0$ – the condition in (i) holds trivially for every $q > 1$. However, if $\mu < 0$ and

$$q \geq q_\mu^* := 1 - \frac{2}{\alpha_-}$$

then equation (P_μ) has no moderate solution except for the trivial solution.

Lemma 3.13. *Let $\mu < C_H(\Omega)$ and put*

$$q_{\mu,c} = \frac{N+1-\alpha_-}{N-1-\alpha_-}.$$

Then, for $y \in \partial\Omega$,

$$K_\mu^\Omega(\cdot, y) \in L^q(\Omega, \delta^{\alpha_+}) \iff q < q_{\mu,c}.$$

For every $q \in (1, q_{\mu,c})$ there exists a number $c = c(q, N, \mu)$ such that

$$(3.15) \quad \|K_\mu^\Omega[\nu]\|_{L^{\frac{N+\alpha_+}{N-1-\alpha_-}}(\Omega, \delta^{\alpha_+})} \leq c \|\nu\| \quad \forall \nu \in \mathfrak{M}(\partial\Omega).$$

Proof. Recall that

$$(3.16) \quad K_\mu^\Omega(x, y) \sim |x-y|^{2-N-\alpha_+} (\delta(x)/|x-y|)^{\alpha_+} = \delta(x)^{\alpha_+} |x-y|^{2\alpha_- - N},$$

(see [10, Section 2.2]). Therefore,

$$c' \left(\frac{\delta(x)}{|x-y|} \right)^{\alpha_+} |x-y|^{1+\alpha_- - N} \leq K_\mu(x, y) \leq c |x-y|^{1+\alpha_- - N}.$$

It follows that $K_\mu(\cdot, y) \in L^q(\Omega, \delta^{\alpha_+})$ if and only if

$$I := \int_0^1 t^{q(1+\alpha_- - N)} t^{\alpha_+} t^{N-1} dt < \infty$$

and

$$\|K_\mu(\cdot, y)\|_{L^q(\Omega, \delta^{\alpha_+})} \sim I.$$

A simple computation shows that $I < \infty$ if and only if

$$q < q_{\mu, c} = \frac{N + 1 - \alpha_-}{N - 1 - \alpha_-}.$$

Finally,

$$\|K_\mu^\Omega[\nu]\|_{L^q(\Omega, \delta^{\alpha_+})} \leq \int_{\partial\Omega} \|K_\mu(\cdot, y)\|_{L^q(\Omega, \delta^{\alpha_+})} d|\nu|(y) \leq c\|\nu\|.$$

□

Corollary 3.14. *Let $\mu < 1/4$. If $1 < q < q_{\mu, c}$ then the boundary value problem (P_μ^ν) has a solution for every Borel measure ν . Moreover, if $q \geq q_{\mu, c}$ then problem (P_μ^ν) has no solution when ν is the Dirac measure.*

Proof. In view of Lemma 3.13, the first assertion follows from Theorem 3.7. The second assertion follows from Proposition 3.6. □

APPENDIX A. NON-UNIQUENESS FOR $C_H(\Omega) < \mu < 1/4$

We are going to show that for $C_H(\Omega) < \mu < 1/4$ the problem

$$(P_\mu^0) \quad \begin{cases} -\mathcal{L}_\mu u + u^q &= 0 & \text{in } \Omega, \\ \text{tr}_\mu^*(u) &= 0, \end{cases}$$

admits a nontrivial solution. This was proved in [5, Theorem 5.3]. Here we provide a more direct argument.

Recall that if $C_H(\Omega) < 1/4$ then the operator $-\mathcal{L}_{C_H(\Omega)}$ admits a positive ground state solution $\phi_H \in H_0^1(\Omega)$ such that $-\mathcal{L}_{C_H(\Omega)}\phi_H = 0$ in Ω , see [9].

Proposition A.1. *Assume that $C_H(\Omega) < \mu < 1/4$ and $q > 1$. Then (P_μ^0) admits a positive solution U_0 such that*

$$\liminf_{x \rightarrow \partial\Omega} \frac{U_0(x)}{\phi_H(x)} > 0.$$

Proof. Since $-\mathcal{L}_{C_H(\Omega)}\phi_H = 0$ in Ω , for a small $\tau > 0$ we obtain

$$-\mathcal{L}_\mu(\tau\phi_H) + (\tau\phi_H)^q = -\frac{\mu - C_H(\Omega)}{\delta^2}(\tau\phi_H) + (\tau\phi_H)^q \leq 0 \quad \text{in } \Omega,$$

so that $\tau\phi_H$ is a subsolution for (P_μ^0) in Ω .

Fix $\rho \in (0, \bar{\rho}]$. Similarly to the proof of Theorem 3.7, for every $k \geq 0$ denote $\nu_{\rho, k} = kdS_{\Sigma_\rho}$ and let $\nu \in \mathfrak{M}^+(\partial\Omega_\rho)$ be the measure such that $\nu\mathbf{1}_{\partial\Omega} = 0$ and $\nu\mathbf{1}_{\Sigma_\rho} = \nu_{\rho, k}$. By Proposition 3.5 there exists a (unique) solution of $(P_\mu^\nu(\rho))$ with this boundary data. Denote this solution by $U_{0, k}$ and put

$$U_{0, \infty} = \lim_{k \rightarrow \infty} U_{0, k}.$$

Let $R \in (0, \rho)$. By Lemma 3.4 there exists a unique solution v_R of (3.5) in D_R with $f = 2U_{0, \infty}$ on Σ_R . We define,

$$\bar{u} := \min\{U_{0, \infty}, u_R\} \quad \text{in } D_R \cap \Omega_\rho.$$

Then \bar{u} is a supersolution of (P_μ) in $D_R \cap \Omega_\rho$, $\bar{u} = U_{0,\infty}$ in $D_R \cap \Omega_{\rho'}$ for some $\rho' \in (R, \rho)$ and $\bar{u} = u_R$ in $D_{R'} \cap \Omega_\rho$ for some $R' \in (R, \rho')$. Therefore setting $\bar{u} = u_R$ in $\Omega \setminus \Omega_\rho$ and $\bar{u} = U_{0,\infty}$ in $\Omega \setminus D_R$ provides an extension (still denoted by \bar{u}) that is a supersolution of (P_μ) in Ω . As $\bar{u} = U_{0,\infty}$ in a neighborhood of $\partial\Omega$ it follows that $\bar{u} \sim \delta^{\alpha_+}$ in such a neighborhood. On the other hand $\phi_H \sim \delta^{a_+}$ where $a_+ := \frac{1}{2} + \sqrt{\frac{1}{4} - C_H(\Omega)}$. As $C_H(\Omega) < \mu$ it follows that $\alpha_+ < a_+$ so that $\delta^{\alpha_+} > \delta^{a_+}$. Therefore $\tau\phi_H < \bar{u}$ near $\partial\Omega$ and therefore, by Lemma 3.1, everywhere in Ω . Finally by Lemma 3.2 we conclude that there exists a solution U_0 of (P_μ) in Ω such that $\tau\phi_H < U_0 < \bar{u}$. Thus U_0 is a positive solution such that $\text{tr}^*(U_0) = 0$. \square

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